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Quasigroups Connected with Clifford Groups

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In Clifford groups, a nonassociative product is defined which leads to the definition of nonassociative groups. These nonassociative groups have matrix representations on the condition that the "row by column" product of two matrices is replaced by the "column by column" product. A nonassociative group of transformations connected with the Lorentz group is determined, together with its irreducible, double-valued matrix representation, whose matrices undergo the "column by column" product.

1. INTRODUCTION

The idea of applying a nonassociative algebra in physics first appeared with Jordan (1932, 1933). Subsequently Jordan *et al.* (1934) discussed certain linear, real nonassociative algebras which satisfy the ordinary postulates for addition, the commutative law of multiplication, and the distributive law. They demonstrated that, with a single exception, every algebra satisfying the above postulates is equivalent to an algebra M whose elements are real matrices with products $x \cdot y$ defined by the Jordan product $x \cdot y = (xy + yx)/2$, where xy denotes the matrix product. The single exception is the algebra M_3^8 of all three-dimensional Hermitian matrices, with elements in the real nonassociative algebra of Cayley numbers. Albert (1934) proved that M_3^8 is a new algebra which is not equivalent to any algebra obtained by the Jordan product of real matrices. Full expositions of Jordan algebra are given by Braun and Koecher (1966) and Jacobson (1968). A further discussion of the physical aspects of nonassociative algebras was given by Segal (1947) and Sherman (1956). Applications in quantum mechanics and elementary particle physics of the algebra of octonions and of Jordan algebra were discussed by Gürsey (1979). In this paper, we discuss a hitherto unexploited

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approach to nonassociative algebras and groups. From the mathematical standpoint, this investigation belongs to the realm of quasigroups (Pflugfelder, 1990; Chein *et al.*, 1990).

2. CLIFFORD QUASIGROUPS

We consider Clifford algebras with the generators $\gamma_1, \gamma_2, \ldots, \gamma_N$ for $N = 1, 2, \ldots,$

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}, \qquad \mu, \nu = 1, 2, \ldots, N \tag{1}$$

For a fixed N, the basis of a Clifford algebra consists of the identity 1, the generators $\gamma_1, \ldots, \gamma_N$, and all linearly independent products of these generators. The dimension of this algebra is 2^N. Since the respective group G contains the element -1, its order is 2^{N+1}.

Definition 2.1. The involution operation I in a Clifford group G is defined by

$$\begin{gathered} I(\gamma_\mu) = -\gamma_\mu, \qquad I(I(\gamma_\mu)) = \gamma_\mu, \qquad I(\pm 1) = \pm 1 \\ I(\gamma_\mu \gamma_\nu \ldots \gamma_\sigma) = I(\gamma_\sigma) \ldots I(\gamma_\nu) I(\gamma_\mu) \\ \gamma_\mu, \gamma_\nu, \ldots, \gamma_\sigma \in G \end{gathered} \tag{2}$$

For the group elements consisting of products of γ's, we introduce the notation

$$\gamma_\mu \gamma_\nu \ldots \gamma_\sigma = \gamma_{\mu\nu\ldots\sigma} \tag{3}$$

Definition 2.2. Define the following group automorphism by the equality

$$I(\gamma_A)\gamma_B = \gamma_C \qquad \text{for a fixed } \gamma_A \tag{4}$$

where, for brevity, γ_A, γ_B, and γ_C now denote arbitrary elements of a Clifford group, i.e., also arbitrary products of the elements γ_α, $\alpha = 1, 2, \ldots, N$, where for any $\gamma_A = \gamma_\alpha \gamma_\beta \cdots \gamma_\rho$, we have $\gamma_A^{-1} = \gamma_\rho \cdots \gamma_\beta \gamma_\alpha$.

Corollary 2.1. The automorphisms defined in Eqs. (2) and (4) form groups.

Definition 2.3. Define the product

$$(\gamma_{\mu\nu\ldots\sigma}) \cdot (\gamma_{\epsilon\eta\ldots\rho}) := I(\gamma_{\epsilon\eta\ldots\rho})\gamma_{\mu\nu\ldots\sigma} \tag{5}$$

where the product on the right-hand side is the associative product in Clifford groups. For single-γ group elements this definition reads

$$\gamma_\mu \cdot \gamma_\nu := I(\gamma_\nu)\gamma_\mu \tag{6}$$

The following definitions together with the consequences which are implied by them are based on Definition 2.3.

We will write the elements $\gamma_{\mu\nu\ldots\sigma}$ in the "dot" product on the left-hand side of Eq. (5) with a caret, which means that $(\gamma_{\mu\nu\ldots\sigma}) \cdot (\gamma_{\epsilon\eta\ldots\rho})$ will be replaced by $(\hat{\gamma}_{\mu\nu\ldots\sigma}) \cdot (\hat{\gamma}_{\epsilon\eta\ldots\rho})$.

In the following, we will denote the Clifford group elements by a, b, c, . . . , or, alternatively, by $g_1, g_2, \ldots, g_m$.

Definition 2.4. If 1 denotes the unit element in a Clifford group and a an element in that group, the right unit element $\hat{1}$ for the "dot" product in Eq. (5) is defined by the equality

$$\hat{a} \cdot \hat{1} = I(1)a = a \to \hat{a} \qquad \text{for any} \quad a \in G \tag{7}$$

where the arrow on the right-hand side indicates the passage from Clifford group elements to elements undergoing the "dot" product.

Corollary 2.2. The right unit element is not at the same time a left unit element since from Eq. (5) we obtain

$$\hat{1} \cdot \hat{a} = I(a)1 = I(a) \tag{8}$$

and in a Clifford group, in a general case, we have $I(a) \neq a$.

Corollary 2.3. The right unit element $\hat{1}$ applied from the left twice does not change any element in the "dot" product since we have

$$\hat{1} \cdot (\hat{1} \cdot \hat{a}) = I[I(a)1]1 = a \to \hat{a} \qquad \text{for any} \quad a \in G \tag{9}$$

Corollary 2.4. From Eq. (5) it follows that

$$(\hat{a} \cdot \hat{b}) \cdot \hat{c} = \hat{a} \cdot [\hat{c} \cdot (\hat{1} \cdot \hat{b})] \qquad \text{for any} \quad a, b, c \in G \tag{10}$$

Proof. We have $(\hat{a} \cdot \hat{b}) \cdot \hat{c} = I(c)\,(\hat{a} \cdot \hat{b}) = I(c)I(b)a$ and $\hat{a} \cdot [\,\hat{c} \cdot (\hat{1} \cdot \hat{b})] = I[\hat{c} \cdot (\hat{1} \cdot \hat{b})]a = I[I(\hat{1} \cdot \hat{b})c]a = I(c)I(b)a$.

Corollary 2.5. The "dot" product is nonassociative. Equation (10) replaces the law of associativity of multiplication in a Clifford group.

Definition 2.5. The nonassociative multiplication in Eq. (5) is performed from left to right:

$$\hat{a} \cdot \hat{b} \cdot \hat{c} \cdot \ldots \cdot \hat{z} = \{[(\hat{a} \cdot \hat{b}) \cdot \hat{c}\,] \cdot \ldots \cdot \hat{z}\} \tag{11}$$

Corollary 2.6. From Eq. (5) it follows that

$$ab = \hat{b} \cdot I(a) \qquad \text{for any} \quad a, b \in G \tag{12}$$

where on the right-hand side $I(a) = a' \to \hat{a}'$.

Definition 2.6. If a^{-1} is the inverse of a in the Clifford group, then the right inverse of $\hat{a}$ is defined by

$$\hat{a}^{-1} = I(a^{-1}) \tag{13}$$

since from Eq. (12) we obtain $1 = a^{-1}\, a = \hat{a} \cdot I(a^{-1}) \to \hat{1}$.

Corollary 2.7. $\hat{a}^{-1} = I(a^{-1})$ is equal to the left inverse since from Eqs. (10) and (9) we have

$$\hat{1} = \hat{1} \cdot \hat{1} = \hat{1} \cdot (\hat{a} \cdot \hat{a}^{-1}) = [\hat{1} \cdot (\hat{1} \cdot \hat{a}^{-1})] \cdot \hat{a} = \hat{a}^{-1} \cdot \hat{a} \tag{14}$$

Corollary 2.8. From the definitions in Eqs. (5), (7), (11), and (13), it follows that the γ's in the "dot" product form a nonassociative group. Consequently, these equations define a nonassociative group connected with a Clifford group. The respective nonassociative Clifford algebra is defined to have a basis consisiting of the right identity $\hat{1}$, the generators $\hat{\gamma}_\mu$, $\mu = 1, \ldots, N$, fulfilling the condition $\hat{\gamma}_\mu \cdot \hat{\gamma}_\nu + \hat{\gamma}_\nu \cdot \hat{\gamma}_\mu = -2\delta_{\mu\nu}\,\hat{1}$, $\mu, \nu = 1, \ldots, N$, and all linearly independent products of these generators. The dimension of this algebra is 2^N. To perform the multiplication of these generators or their products, Eq. (5) has to be applied. An example is given in Section 5.

In order to avoid unnecessary brackets in the formulas, we shall omit from now on the "dot" while multiplying any element from the left or from the right by the right unit element. Instead of $\hat{1} \cdot \hat{a}$ or $\hat{a} \cdot \hat{1}$, we shall write $\hat{1}\hat{a}$ or $\hat{a}\hat{1}$, respectively; instead of $\hat{b} \cdot (\hat{1} \cdot \hat{a})$, we shall write $\hat{b} \cdot \hat{1}\hat{a}$; and $\hat{a} \cdot \hat{1} \cdot \hat{b}$ will be replaced by $\hat{a}\hat{1} \cdot \hat{b}$. Consequently, Eq. (10) takes the form

$$(\hat{a} \cdot \hat{b}) \cdot \hat{c} = \hat{a} \cdot (\hat{c} \cdot \hat{1}\hat{b}) \tag{15}$$

Corollary 2.9. From Eq. (15) it follows that

$$\hat{1}(\hat{a} \cdot \hat{b}) = \hat{b} \cdot \hat{a} \tag{16}$$

Corollary 2.10. From $\hat{a} \cdot \hat{a}^{-1} = \hat{1}$, it follows that $(\hat{1}\hat{a}) \cdot (\hat{1}\hat{a}^{-1}) = \hat{1}$.

Proof. We utilize Eqs. (9) and (10). Writing $\hat{a}^{-1} = \hat{b}$ and multiplying $\hat{a} \cdot \hat{b} = \hat{1}$ from the right by $(\hat{1}\hat{b})^{-1}$, we obtain $(\hat{a} \cdot \hat{b}) \cdot (\hat{1}\hat{b})^{-1} = \hat{a} \cdot [(\hat{1}\hat{b})^{-1} \cdot \hat{1}\hat{b}] = \hat{1}(\hat{1}\hat{b})^{-1} = \hat{a}$. Multiplying $\hat{1}(\hat{1}\hat{b})^{-1} = \hat{a}$ by $\hat{1}$ from the left, we obtain $(\hat{1}\hat{b})^{-1} = \hat{1}\hat{a}$, which multiplied by $\hat{1}\hat{b}$ from the right yields the equalities.

$$(\hat{1}\hat{b})^{-1} \cdot (\hat{1}\hat{b}) = (\hat{1}\hat{a}) \cdot (\hat{1}\hat{b}) = (\hat{1}\hat{a}) \cdot (\hat{1}\hat{a}^{-1}) = \hat{1} \tag{17}$$

Corollary 2.11. We have

$$(\hat{a} \cdot \hat{b})^{-1} = \hat{a}^{-1} \cdot \hat{b}^{-1} \tag{18}$$

Proof. From Eqs. (10) and (17) we obtain

$$\begin{aligned}(\hat{a} \cdot \hat{b}) \cdot (\hat{a}^{-1} \cdot \hat{b}^{-1}) &= \hat{a} \cdot [(\hat{a}^{-1} \cdot \hat{b}^{-1}) \cdot \hat{1}\hat{b}] \\ &= \hat{a} \cdot [\hat{a}^{-1} \cdot (\hat{1}\hat{b} \cdot \hat{1}\hat{b}^{-1})] = \hat{a} \cdot \hat{a}^{-1} = \hat{1}\end{aligned} \tag{19}$$

Corollary 2.12. From Eq. (18), it follows that $(\hat{1}\hat{a})^{-1} = \hat{1}\hat{a}^{-1}$, since $\hat{1} = \hat{1}^{-1}$.

Corollary 2.13. For p factors $\hat{g}_1 \cdot \hat{g}_2 \cdot \ldots \cdot \hat{g}_p$, we have

$$(\hat{g}_1 \cdot \hat{g}_2 \cdot \ldots \cdot \hat{g}_p)^{-1} = g_1^{-1} \cdot g_2^{-1} \cdot \ldots \cdot g_p^{-1} \tag{20}$$

since from Eq. (18) and from $(\hat{g}_1 \cdot \hat{g}_2 \cdot \ldots \cdot \hat{g}_k)^{-1} = g_1^{-1} \cdot g_2^{-1} \cdot \ldots \cdot g_k^{-1}$, it follows that $(g_1 \cdot g_2 \cdot \ldots \cdot g_k \cdot g_{k+1})^{-1} = (g_1 \cdot g_2 \cdot \ldots \cdot g_k)^{-1} \cdot g_{k+1}^{-1} = g_1^{-1} \cdot g_2^{-1} \cdot \ldots \cdot g_k^{-1} \cdot g_{k+1}^{-1}$.

Corollary 2.14. From Eqs. (16) and (15), it follows that

$$\hat{1}(\hat{a} \cdot \hat{b} \cdot \hat{c}) = \hat{c} \cdot \hat{1}\hat{b} \cdot \hat{a} \tag{21}$$

since we have $\hat{1}[(\hat{a} \cdot \hat{b}) \cdot \hat{c}] = \hat{c} \cdot (\hat{a} \cdot \hat{b})$ and $\hat{c} \cdot (\hat{a} \cdot \hat{b}) = \hat{c} \cdot \hat{1}\hat{b} \cdot \hat{a}$.

Corollary 2.15. For p factors $\hat{g}_1 \cdot \hat{g}_2 \cdot \ldots \cdot \hat{g}_p$, we have

$$\hat{1}(\hat{g}_1 \cdot \hat{g}_2 \cdot \ldots \cdot \hat{g}_{p-1} \cdot \hat{g}_p) = \hat{g}_p \cdot \hat{1}\hat{g}_{p-1} \cdot \ldots \cdot \hat{1}\hat{g}_2 \cdot \hat{g}_1 \tag{22}$$

Proof. Let $\hat{g}_1 \cdot \hat{q}_2 \cdot \ldots \cdot \hat{g}_p = (\hat{g}_1 \cdot \hat{q}_2 \cdot \ldots \cdot \hat{g}_{p-2}) \cdot \hat{g}_{p-1} \cdot \hat{g}_p = \hat{W}$, then, owing to Eqs. (15) and (21), we obtain

$$\begin{aligned}
\hat{1}\hat{W} &= \hat{g}_p \cdot \hat{1}\hat{g}_{p-1} \cdot (\hat{g}_1 \cdot \hat{g}_2 \cdot \ldots \cdot \hat{g}_{p-2}) \\
&= (\hat{g}_p \cdot \hat{1}\hat{g}_{p-1}) \cdot [(\hat{g}_1 \cdot \hat{g}_2 \cdot \ldots \cdot \hat{g}_{p-3}) \cdot \hat{g}_{p-2}] \\
&= (\hat{g}_p \cdot \hat{1}\hat{g}_{p-1}) \cdot \hat{1}\hat{g}_{p-2} \cdot (\hat{g}_1 \cdot \hat{g}_2 \cdot \ldots \cdot \hat{g}_{p-3}) \\
&= (\hat{g}_p \cdot \hat{1}\hat{g}_{p-1} \cdot \hat{1}\hat{g}_{p-2}) \cdot [(\hat{g}_1 \cdot \hat{g}_2 \cdot \ldots \cdot \hat{g}_{p-4}) \cdot \hat{g}_{p-3}]
\end{aligned} \tag{23}$$

and repeating this procedure, we obtain Eq. (22).

Observation 2.1. If the nonassociative product definition in Eq. (5) is replaced by the definition

$$\hat{a} \cdot \hat{b} := aI(b) \tag{24}$$

we again obtain Eqs. (7), (11), and (13), which define a nonassociative group connected with a Clifford group. The choice between product definitions in Eqs. (5) and (24) will be made in the next section.

Corollary 2.16. The definitions in Eqs. (5) or (24) of a nonassociative product together with Eqs. (7), (11), and (13) hold for any group G, with elements g and unit element e, for which an involution operation I exists, which fulfils the conditions $I(g) \in G$, $I[I(g)] = g$, $I(g_1 g_2 \ldots g_p) = I(g_p) \ldots I(g_2)I(g_1)$, $I(e) = e$ for any $g, g_1, g_2, \ldots, g_p \in G$.

3. REPRESENTATIONS OF NONASSOCIATIVE GROUPS

The product definition in Eq. (5) together with the conditions (7)–(13) connected with it are fulfilled by square, nonsingular matrices, for which the "row by column" multiplication is replaced by the "column by column" multiplication (or, alternatively, "row by row" multiplication). This type of product of two matrices was introduced by Banachiewicz (1929, 1937, 1938, 1959) and matrices undergoing this product rule were called by him "cracovians." An exposition of cracovian algebra was given by Sierpiński (1951).

Consequently, a square or rectangular table of numbers or other symbols can be called a matrix or a cracovian, depending on the product definition of two such tables. This means that representations of nonassociative Clifford groups exist.

Definition 3.1. A representation by linear substitutions of a nonassociative group G' connected with a Clifford group G is a cracovian quasigroup onto which the nonassociative group is homomorphic. It consists of the assignment of a quadratic cracovian $C(\hat{a})$ to each nonassociative group element $\hat{a}$ in such a way that

$$C(\hat{a}) \cdot C(\hat{b}) = C(\hat{a} \cdot \hat{b}) \qquad \text{for all} \quad \hat{a}, \hat{b} \in G' \tag{25}$$

The notion of irreducibility of a matrix representation carries over for a cracovian representation. The criteria of irreducibility of cracovian representations have to be determined, however, since the irreducibility criteria of matrix representations of groups do not carry over for cracovian representations of nonassociative groups connected with Clifford groups. In particular, this means that a set of square tables of complex numbers in its quality of being a matrix representation of a group can be a reducible representation, while in its quality of being a cracovian representation of the respective nonassociative group, it can be an irreducible cracovian representation. An example of such a case will be given in Section 5.

Observation 3.1. The notions of faithful or unfaithful matrix representations and of their dimension in the case of groups carry over for cracovian representations of nonassociative groups connected with Clifford groups. Each cracovian quasigroup is its own faithful representation.

4. THE NONASSOCIATIVE GROUP OF TRANSFORMATIONS CONNECTED WITH THE LORENTZ GROUP

Writing a four-vector in the Minkowski space in the form

$$\vec{x} = \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 + \gamma_4 x_4 \tag{26}$$

with $x_4 = ict$, we can write its transformation under Lorentz rotations in the form

$$\vec{x}' = \gamma_\mu x'_\mu = S^{-1}(\gamma_\mu x_\mu)S \tag{27}$$

where S and S^{-1} are biquaternion transformations (Sommerfeld, 1944). We have

$$S = A\gamma_{23} + B\gamma_{31} + C\gamma_{12} + D + ia\gamma_{14} + ib\gamma_{24} + ic\gamma_{34} - id\gamma_5 \tag{28}$$

where $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, and reversing on the right-hand side the signs of A, B, C, a, b, and c, we obtain S^{-1}. With real Cayley–Klein parameters A, B, C, D, a, b, c, and d, we obtain $SS^{-1} = S^{-1}S = 1$ on the two conditions:

$$A^2 + B^2 + C^2 + D^2 - a^2 - b^2 - c^2 - d^2 = 1 \tag{29}$$

$$Aa + Bb + Cc + Dd = 0 \tag{30}$$

From Eq. (27), we obtain the rotation matrix $\mathcal{A}_m$ with the elements

$$\begin{aligned}
&a_{11} = (D^2 + A^2 - B^2 - C^2) + (d^2 + a^2 - b^2 - c^2)\\
&a_{22} = (D^2 - A^2 + B^2 - C^2) + (d^2 - a^2 + b^2 - c^2)\\
&a_{33} = (D^2 - A^2 - B^2 + C^2) + (d^2 - a^2 - b^2 + c^2)\\
&a_{44} = (D^2 + A^2 + B^2 + C^2) + (d^2 + a^2 + b^2 + c^2)\\
&a_{12} = 2(AB + CD) + 2(ab + cd), \qquad a_{21} = 2(AB - CD) + 2(ab - cd)\\
&a_{13} = 2(AC - BD) + 2(ac - bd), \qquad a_{23} = 2(BC + AD) + 2(bc + ad)\\
&a_{14} = 2i[(Da - Ad) - (Bc - Cb)], \qquad a_{24} = 2i[(Ac - Bd) - (Ca - Db)]\\
&a_{31} = 2(AC + BD) + 2(ac + bd), \qquad a_{41} = 2i[(Ad + Cb) - (Bc + Da)]\\
&a_{32} = 2(BC - AD) + 2(bc - ad), \qquad a_{42} = 2i[(Ac + Bd) - (Ca + Db)]\\
&a_{34} = 2i[(Ba - Ab) - (Cd - Dc)], \qquad a_{43} = 2i[(Ba - Ab) - (Dc - Cd)]
\end{aligned} \tag{31}$$

We now rewrite the Lorentz rotations in Eq. (27) in the matrix-product form

$$x'_{\mathrm{m}} = \mathcal{A}_{\mathrm{m}} x_{\mathrm{m}} \tag{32}$$

and in the respective cracovian-product form

$$x'_{\mathrm{c}} = x_{\mathrm{c}} \cdot T\mathcal{A}_{\mathrm{c}} = x_{\mathrm{c}} \cdot P_{\mathrm{c}} \tag{33}$$

where the column matrices x'_{m} and x_{m} in Eq. (32) are identical with the column cracovians x'_{c} and x_{c} in Eq. (33), where T denotes the "transpose" cracovian [see Eq. (6.3)] and the square matrix $\mathcal{A}_{\mathrm{m}}$ in Eq. (32) is identical with the square cracovian $\mathcal{A}_{\mathrm{c}}$ in Eq. (33). The subscripts c and m have been introduced to distinguish the cracovian tables from the matrix tables.

If the index j of the matrix elements a_{jk} in Eq. (31) is identified with the column index of a cracovian and the index k with its row index, then the elements a_{jk} in Eq. (31) are the elements of the cracovian $P_c = T\mathcal{A}_c$ in Eq. (33). This is due to the fact that the cracovian T transposes the cracovian $\mathcal{A}_c$ (see Corollary 6.3) The cracovian P_c may, for brevity, be called a quasi-rotation cracovian, since it represents nonassociative transformations connected with Lorentz rotations.

5. AN IRREDUCIBLE DOUBLE-VALUED CRACOVIAN REPRESENTATION

We will determine an irreducible double-valued cracovian representation of the quasi-rotations P_c in Eq. (33). We first consider the nonassociative biquaternion group, whose elements are determined from Eq. (5). For example, we obtain

$$\hat{\gamma}_{23} \cdot \hat{\gamma}_{31} = I(\gamma_{31})\gamma_{23} = \gamma_{13}\gamma_{23} = -\gamma_{12} = \gamma_{21}$$

which is identified with $\hat{\gamma}_{21}$, and

$$\hat{\gamma}_{23} \cdot i\hat{\gamma}_{14} = iI(\gamma_{14})\gamma_{23} = i\gamma_{41}\gamma_{23} = -i\gamma_{1234} = -i\gamma_5$$

which is identified with $-i\hat{\gamma}_5$. The resulting Cayley table is given in Table I. The two inequivalent two-dimensional irreducible representations (irreps) of the biquaternion group at the same time are cracovian irreps of the respective nonassociative biquaternion group. The nonassociative biquaternion group has, however, another cracovian irrep of the form

Table I. Multiplication Table for Nonassociative Biquaternion Group with $\hat{1}$ Denoting the Right Unit Element and Numbers Denoting the Indices of the Respective $\hat{\gamma}$-Symbols

	$\hat{1}$	23	31	12	$i14$	$i24$	$i34$	$i5$
$\hat{1}$	$\hat{1}$	32	13	21	$i41$	$i42$	$i43$	$i5$
23	23	$\hat{1}$	21	31	$-i5$	$i43$	$i24$	$i41$
31	31	12	$\hat{1}$	32	$i34$	$-i5$	$i41$	$i42$
12	12	13	23	$\hat{1}$	$i42$	$i14$	$-i5$	$i43$
$i14$	$i14$	$-i5$	$i43$	$i24$	$-\hat{1}$	12	13	23
$i24$	$i24$	$i34$	$-i5$	$i41$	21	$-\hat{1}$	23	31
$i34$	$i34$	$i42$	$i14$	$-i5$	31	32	$-\hat{1}$	12
$i5$	$i5$	$i14$	$i24$	$i34$	32	13	21	$-\hat{1}$

$$L(\hat{\gamma}_{23}) = \begin{Bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{Bmatrix}, \qquad M(\hat{\gamma}_{31}) = \begin{Bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{Bmatrix}$$

$$N(\hat{\gamma}_{21}) = \begin{Bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{Bmatrix}, \qquad T(\hat{1}) = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix} \tag{34}$$

$$l(i\hat{\gamma}_{14}) = iL, \qquad m(i\hat{\gamma}_{24}) = iM, \qquad n(i\hat{\gamma}_{34}) = -iN, \qquad t(i\hat{\gamma}_5) = -iT$$

where here and in the following we use wavy brackets for cracovian tables to distinguish them from matrix tables. To prove the irreducibility of the representation in Eq. (34), we consider the nonassociative quaternion group in Table I, consisting of the four elements $\hat{1}$, $\hat{\gamma}_{23}$, $\hat{\gamma}_{31}$, $\hat{\gamma}_{12}$ together with the same elements multiplied by -1. This has four one-dimensional representations given in Table II, and no other one-dimensional representations exist. If the cracovians in Eq. (34) were reducible to a block-diagonal form with one-dimensional blocks and three-dimensional blocks, the one-dimensional blocks would be identical with one of the four representations in Table II. This, however, is impossible since the four-dimensional cracovian $-T$ cannot be changed into the cracovian T in Eq. (34) by any similarity transformation. Consequently, three-dimensional blocks also are excluded. A similarity transformation [see Eq. (6.17)] cannot lead to two-dimensional-block cracovians since the blocks connected with L, M, and N would be of the form

$$\begin{Bmatrix} k + ir & s + it \\ p + iq & u + iv \end{Bmatrix} \tag{35}$$

with real k, r, s, t, p, q, u, and v. The reduced cracovians would have to obey the Cayley table. From Table I, we then would find that $k + ir = u + iv = 0$, and $s + it = -(p + iq)$. From $L^2 = M^2 = N^2 = T$, it follows that

Table II. The Four One-Dimensional Representations of the Nonassociative Quaternion Group

	T	$-T$	L	$-L$	M	$-M$	N	$-N$
(1)	1	1	1	1	1	1	1	1
(2)	1	1	−1	−1	−1	−1	1	1
(3)	1	1	1	1	−1	−1	−1	−1
(4)	1	1	−1	−1	1	1	−1	−1

$p = 1$ and $q = 0$, and therefore the equality $L \cdot M = N$ which is required by Table I cannot be fulfilled by the block-diagonal cracovians. Consequently, the cracovian representation in Eq. (34) is irreducible. This means that Burnside's theorem for matrix representations of groups does not hold for cracovian representations of nonassociative groups.

We now consider the nonassociative biquaternion transformation

$$\hat{Q} = A\hat{\gamma}_{23} + B\hat{\gamma}_{31} + C\hat{\gamma}_{21} + D\hat{1} + ai\hat{\gamma}_{14} + bi\hat{\gamma}_{24} + ci\hat{\gamma}_{43} - di\hat{\gamma}_5 \tag{36}$$

in which we replace the nonassociative biquaternion elements by their cracovian representation in Eq. (34), thus obtaining the cracovian

$$Q_c = \begin{Bmatrix} (D + id) & -(A + ia) & -(B + ib) & -(C + ic) \\ (A + ia) & (D + id) & -(C + ic) & (B + ib) \\ (B + ib) & (C + ic) & (D + id) & -(A + ia) \\ (C + ic) & -(B + ib) & (A + ia) & (D + id) \end{Bmatrix} \tag{37}$$

with the parameters D, A, B, C, d, a, b, and c satisfying Eqs. (29) and (30). Owing to the irreducibility of the representation in Eq. (34), this cracovian is irreducible. We have $Q_c \cdot Q_c = T$, which means that $Q_c = Q_c^{-1}$. The cracovian Q_c with $a = b = c = d = 0$ appears in Banachiewicz (1938).

The cracovian Q_c yields a double-valued representation of the quasi-rotations P_c in Eq. (33). The following proof is analogous to that of Wigner (1959) concerning the two-to-one homomorphism of the group $SU(2)$ onto the group $SO(3)$.

The square cracovian form X_c of the vector $\vec{x}_c$ connected with the column cracovian x_c in Eq. (33) in the basis (L, M, N, T) in Eq. (34) is given by

$$X_c = \begin{Bmatrix} x_4 & -x_1 & -x_2 & -x_3 \\ x_1 & x_4 & -x_3 & x_2 \\ x_2 & x_3 & x_4 & -x_1 \\ x_3 & -x_2 & x_1 & x_4 \end{Bmatrix} = (\vec{x}_c, \vec{q}_c) \tag{38}$$

where $(\vec{x}_c, \vec{q}_c)$ denotes a scalar product of $\vec{x}_c$ and $\vec{q}_c = (L, M, N, T)$.

We now consider the nonassociative transformation

$$Q_c \cdot (TX_c) \cdot Q_c^* = X_c' = (\vec{x}_c', \vec{q}_c) \tag{39}$$

where * denotes the conjugate complex operation. It can be verified that the coordinates x_1', x_2', x_3', x_4' calculated from Eq. (39) are those calculated from Eqs. (33) with the elements of the cracovian P_c in Eqs. (31) in which the first index of a_{jk} is to be interpreted as the column index of P_c. This means that the transformation in Eq. (33) which carries x_c into $x_c' = x_c \cdot P_c$ can be determined from Eq. (39). We observe that from Eq. (39), it follows that $(\vec{x}_c')^2 = \vec{x}_c^2$.

To demonstrate that to the product $Q_c^{(1)} \cdot Q_c^{(2)}$ of two transformations in Eq. (37) corresponds the product of two quasi-rotations $P_c(Q_c^{(1)}) \cdot P_c(Q_c^{(2)}) = P_c(Q_c^{(1)} \cdot Q_c^{(2)})$, we write for the first transformation $Q_c^{(1)}$, utilizing Eq. (38),

$$Q_c^{(1)} \cdot T\,(\vec{x}_c, \vec{q}_c) \cdot (Q_c^{(1)})^* = (\vec{x}_c \cdot P_c, \vec{q}_c) \tag{40}$$

and for the successive second transformation $Q_c^{(2)}$ we have

$$Q_c^{(2)} \cdot T[Q_c^{(1)} \cdot T\,(\vec{x}_c, \vec{q}_c) \cdot (Q_c^{(1)})^*] \cdot (Q_c^{(2)})^* = ([\vec{x}_c \cdot P_c(Q_c^{(1)})] \cdot P_c(Q_c^{(2)}), \vec{q}_c) \tag{41}$$

By the repeated application of equality (A.6), the left-hand side of Eq. (41) can be transformed to the form

$$(Q_c^{(2)} \cdot TQ_c^{(1)}) \cdot T(\vec{x}_c, \vec{q}_c) \cdot (Q_c^{(2)} \cdot TQ_c^{(1)})^*$$

By a single application of that equality, the right-hand side takes the form

$$(\vec{x}_c \cdot [P_c(Q_c^{(2)}) \cdot TP_c(Q_c^{(1)})], \vec{q}_c) = (\vec{x}_c \cdot P_c(Q_c^{(2)} \cdot TQ_c^{(1)}), \vec{q}_c)$$

since $TP_c(Q_c^{(1)}) = P_c(TQ_c^{(1)})$. This proves the homomorphism between the nonassociative group of four-dimensional cracovians in Eq. (37) and the nonassociative group of quasi-rotations P_c in Eq. (33). Since the two four-dimensional cracovians T and $-T$ belonging to the cracovian Q_c, and only these two, correspond to the right unit element of the nonassociative group of quasi-rotations, that homomorphism is two to one. If the square table Q_c in Eq. (37) is identified with a matrix, it is reducible with the help of the matrix

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & -i & -1 & 0 \\ 0 & -i & 1 & 0 \\ 1 & 0 & 0 & -i \end{bmatrix} \tag{42}$$

6. RESULTS BASED ON BANACHIEWICZ AND SIERPIŃSKI

This section is based on Banachiewicz (1959) and Sierpiński (1951).

Definition 6.1. A rectangular cracovian is defined as the table of elements

$$A = \begin{Bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{Bmatrix} \tag{6.1}$$

where the first index of an element a_{kl} denotes a column and the second index − a row, and where wavy brackets are used to distinguish a cracovian table from a matrix table.

Observation 6.1. The definitions of a symmetric or antisymmetric matrix carry over from matrices on cracovians.

Definition 6.2. The product of two cracovians A and B, denoted by $A \cdot B$, is obtained by multiplying the columns of A by the columns of B. Two cracovians can therefore be multiplied if they have the same number of rows. The element in the kth column and in the lth row of the cracovian $A \cdot B$ is obtained by multiplying the kth column of A by the lth column of B:

$$(A \cdot B)_{kl} = \sum_i a_{ki} b_{li} \tag{6.2}$$

Observation 6.2. The cracovian product can also be defined in the terms of row by row multiplication.

Corollary 6.1. The multiplication of cracovians is noncommutative.

Definition 6.3. The square diagonal cracovian defined by

$$(T)_{kj} = 1 \quad \text{for } k = \text{j} \qquad \text{and} \qquad (T)_{kj} = 0 \quad \text{for } k \neq j \tag{6.3}$$

corresponds to the right identity in a nonassociative Clifford group. It is called the transpose.

Corollary 6.2. Any cracovian A multiplied by T from the right remains unchanged, and multiplied from the left is changed to the transposed cracovian.

Corollary 6.3. For any cracovian A, we have

$$T(TA) = A \tag{6.4}$$

Corollary 6.4. For any two cracovians A and B, we have

$$T(A \cdot B) = B \cdot A \tag{6.5}$$

Corollary 6.5. For cracovians, the law of associativity of multiplication is replaced by the equality

$$(A \cdot B) \cdot C = A \cdot [C \cdot (TB)] \tag{6.6}$$

Definition 6.4. The multiplication of cracovians is performed from left to right:

$$A \cdot B \cdot C \cdot D \cdot \ldots \cdot Z = \{[A \cdot B) \cdot C] \cdot D \cdot \ldots \cdot Z\} \tag{6.7}$$

Corollary 6.6. For three cracovians A, B, and C, we have

$$T(A \cdot B \cdot C) = C \cdot (TB) \cdot A \tag{6.8}$$

Definition 6.5. The right inverse of a square cracovian A is defined as such a cracovian A^{-1} that we have

$$A \cdot A^{-1} = T \tag{6.9}$$

Corollary 6.7. From Eqs. (6.5) and (6.9), it follows that A^{-1} is also the left inverse.

Corollary 6.8. For a square cracovian A with the inverse A^{-1}, we have

$$(TA) \cdot (TA^{-1}) = T \tag{6.10}$$

Corollary 6.9. For the product of two square cracovians A_1 and A_2 having inverses, we have

$$(A_1 \cdot A_2)^{-1} = A_1^{-1} \cdot A_2^{-1} \tag{6.11}$$

Corollary 6.10. From Eq. (6.11), we have

$$(TA)^{-1} = TA^{-1} \tag{6.12}$$

Observation 6.3. The relation between the matrix product AB and the cracovian product of two tables A and B is given by the equality

$$AB = B \cdot (TA) \tag{6.13}$$

where on the right-hand side, the tables A and B are treated as cracovians.

Observation 6.12. The relation between the matrix product of three tables A, B, and C and the respective cracovian product is given by

$$ABC = C \cdot (TB) \cdot (TA) \tag{6.14}$$

where on the right-hand side, the three tables are treated as cracovians.

Observation 6.13. A square table has the cracovian inverse if and only if it has the matrix inverse.

Observation 6.14. A matrix inverse of a square table is equal to the transpose of the cracovian inverse of that table.

Definition 6.6. Let $\vec{\mathrm{e}}_m$ and $\vec{\mathrm{e}}_c$ denote the column matrix or column cracovian, respectively, constructed from the basis vectors $\vec{\mathrm{e}}_1, \ldots, \vec{\mathrm{e}}_n$ of an n-dimensional linear vector space. Let S_m and S_c be the respective matrix or cracovian transformation of the basis vectors. The change of basis then is defined by the expression

$$\vec{\mathrm{e}}'_{\mathrm{m}} = \tilde{S}_{\mathrm{m}}\vec{\mathrm{e}}_{\mathrm{m}} = \vec{\mathrm{e}}_{\mathrm{c}} \cdot T(TS_{\mathrm{c}}) = \vec{\mathrm{e}}_{\mathrm{c}} \cdot S_{\mathrm{c}} = \vec{\mathrm{e}}'_{\mathrm{c}} \tag{6.15}$$

where $\tilde{S}_{\mathrm{m}}$ denotes the transposed matrix.

Corollary 6.11. Let x_m and x_c (or y_m and y_c) denote the column matrix or column cracovian constructed from the components $x_1, \ldots, x_n$ (or $y_1, \ldots, y_n$) of a vector, referred to the two bases in Eq. (6.15), respectively. The relation between the two sets of components is given by

$$x_m = S_m y_m = y_c \cdot TS_c = x_c \tag{6.16}$$

where the transformation table S has been identified with a matrix S_m or with a cracovian S_c, depending on the type of the employed product.

Corollary 6.12. The relation between matrix and cracovian transformations A_m and B_m or A_c and B_c, respectively, determining the same linear mapping referred to two bases connected according to Eq. (6.15) is given by

$$B_m = S_m^{-1} A_m S_m = S_c \cdot TA_c \cdot S_c^{-1} = B_c \tag{6.17}$$

Corollary 6.13. Let x_m and x_c denote the column matrix and column cracovian, respectively, constructed from the components $x_1, \ldots, x_n$ of a vector. A linear mapping of that vector expressed in the terms of matrix or cracovian product is given by

$$x'_m = A_m x_m = x_c \cdot TA_c = x'_c$$

where the table A has been identified with a matrix A_m or with a cracovian A_c, depending on the type of the employed product.

7. CONCLUSIONS AND DISCUSSION

Nonassociative groups connected with Clifford groups have been defined. That definition carries over on any group in which an involution operation exists. Representations of these nonassociative groups exist in the form of cracovians. An irreducible double-valued cracovian representation of the quasi-rotations P_c connected with Lorentz rotations has been determined. It can be shown that the Dirac equation is covariant with respect to the quasi-rotations P_c. This suggests that the nonassociative group of quasi-rotations P_c may perhaps describe a hidden symmetry of the Dirac equation in the sense of Weyl (1952). The cracovian Q_c could then be considered for description of some dynamical properties of spin-1/2 particles.

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